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An axisymmetric external crack problem for an infinite medium with a cylindrical inclusion*

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SUMMARY

This paper deals with the determination of stresses in an infinite medium containing an external crack surrounding a cylindrical inclusion. The two media are assumed to be homogeneous, isotropic and elastic but with different elastic constants. The continuity of stresses and displacements is assumed at the common cylindrical surface due to perfect bonding. The problem is reduced to the solution of a Fredholm integral equation of the second kind. A closed-form expression is obtained for the stress-intensity factor. The integral equation is solved numerically and the results are used to obtain the numerical values of the stress-intensity factor which are displayed graphically.

1. Introduction

In recent years considerable effort has been devoted to the problem of calculating stresses in infinite and semi-infinite solids, thick plates and infinitely long cylinders containing penny-shaped or external cracks. Adequate references to this type of work may be found in Sneddon and Lowengrub [1] and Kassir and Sih [2]. Recently Srivastav and Lee [3] solved axisymmetric crack problems for media with a cylindrical cavity.

In this paper we discuss the problem of determination of stresses in an infinite medium containing an external crack surrounding a circular cylindrical inclusion. The crack is assumed to be in a plane ($z = 0$) normal to the axis of the cylinder (z-axis). The cylinder is assumed to be in perfect bond with the infinite medium surrounding it. The elastic constants of the two media are assumed to be different. The crack surfaces are subjected to a normal loading. By assuming suitable solutions of the equilibrium equations for the two regions, the problem is reduced to the solution of dual integral equations. The dual integral equations are further reduced to a single Fredholm integral equation of the second kind which is amenable to numerical solution. Solving the Fredholm integral equation numerically, the numerical values of the stress-intensity factor are obtained and then displayed graphically.

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2. Fundamental equations

We denote the region containing the infinite space $b < r < \infty$, $-\infty < z < \infty$ by L_1 and the region containing the cylindrical inclusion $0 < r < b$, $-\infty < z < \infty$ by L_2. For a symmetrical deformation the displacement vector is denoted by $[u_r^i(r,z), 0, u_z^i(r,z)]$ and the stress tensor is denoted by $[\sigma_{rr}^i, \sigma_{\theta\theta}^i, \sigma_{rz}^i, \sigma_{zz}^i]$ and μ_i and ν_i denote the shear modulus and Poisson ratio respectively of the region L_i $(i = 1,2)$. From Sneddon ([4], p. 505), we find that a solution of the equations of elastic equilibrium in the axially symmetric case is given by

$$u_r(r,z) = -\frac{1}{2\mu}\frac{\partial^2 V}{\partial r \partial z}, \quad u_z(r,z) = \frac{1}{2\mu}\left[2(1-\nu)\nabla^2 V - \frac{\partial^2 V}{\partial z^2}\right], \tag{1}$$

where $V(r, z)$ is an axisymmetric biharmonic function. The stress components can easily be determined from the stress-strain relations. We have

$$\sigma_{rr}(r,z) = \frac{\partial}{\partial z}\left[\nu\nabla^2 V - \frac{\partial^2 V}{\partial r^2}\right], \quad \sigma_{rz}(r,z) = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 V - \frac{\partial^2 V}{\partial z^2}\right], \tag{2}$$

$$\sigma_{\theta\theta}(r,z) = \frac{\partial}{\partial z}\left[\nu\nabla^2 V - \frac{1}{r}\frac{\partial V}{\partial r}\right], \quad \sigma_{zz}(r,z) = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 V - \frac{\partial^2 V}{\partial z^2}\right],$$

where μ and ν are shear modulus and Poisson ratio respectively.

A suitable biharmonic function for the region L_1 can be taken as

$$\begin{aligned} V_1(r,z) = &-2\mu_1\int_0^\infty s^{-3}F(s)[2\nu_1 + sz]e^{-sz}J_0(sr)ds \\ &-2\mu_1\int_0^\infty s^{-2}[\{A(s) - 4(1-\nu_1)B(s)\}K_0(sr) - srB(s)K_1(sr)]\sin(sz)\,ds \end{aligned} \tag{3}$$

where $F(s)$, $A(s)$ and $B(s)$ are unknown functions to be determined and $J_p(\ )$ and $K_p(\ )$ denote the Bessel function of the first kind and the modified Bessel function of the second kind respectively and of order $p \geqslant 0$. The components of the displacement vector and of the stress tensor for the region L_1 can be obtained with the help of equations (1), (2) and (3) as follows:

$$\begin{aligned} u_r^1(r,z) = &\int_0^\infty \frac{1}{s}(sz + 2\nu_1 - 1)F(s)J_1(rs)e^{-sz}ds \\ &+\int_0^\infty \{-K_1(rs)A(s) + [4(1-\nu_1)K_1(rs) + rsK_0(rs)]B(s)\}\cos(sz)ds, \end{aligned}$$

$$\begin{aligned} u_z^1(r,z) = &\int_0^\infty \frac{1}{s}(2 - 2\nu_1 + sz)F(s)J_0(rs)e^{-sz}ds \\ &+\int_0^\infty [-K_0(rs)A(s) + rsK_1(rs)B(s)]\sin(sz)ds, \end{aligned}$$

$$\frac{\sigma_{rr}^1}{2\mu_1} = \int_0^\infty \left[(-1+sz)J_0(rs) + (1-2\nu_1 - sz)\frac{J_1(rs)}{rs}\right]$$

$$F(s)e^{-sz}\,ds + \frac{1}{r}\int_0^\infty \{[K_1(rs) + rsK_0(rs)]A(s)$$

$$-[(4-4\nu_1 + r^2s^2)K_1(rs) + (3-2\nu_1)rsK_0(rs)]B(s)\}\cos(sz)\,ds,$$

$$\frac{\sigma_{zz}^1}{2\mu_1} = -\int_0^\infty (1+sz)F(s)J_0(rs)e^{-sz}\,ds - \int_0^\infty s\{A(s)K_0(rs)$$

$$+[2\nu_1K_0(rs) - rsK_1(rs)]B(s)\}\cos(sz)\,ds,$$

$$\frac{\sigma_{rz}^1}{2\mu_1} = -z\int_0^\infty sF(s)J_1(rs)e^{-sz}\,ds + \int_0^\infty s\{K_1(rs)A(s)$$

$$-[2(1-\nu_1)K_1(rs) + rsK_0(rs)]B(s)\}\sin(sz)\,ds. \tag{4}$$

Making use of the following biharmonic function (see [1], p. 195) for the region L_2

$$V_2(r,z) = -2\mu_2\int_0^\infty s^{-2}\{[C(s)+4(1-\nu_1)D(s)]I_0(sr) - srD(s)I_1(sr)\}\sin(sz)\,ds, \tag{5}$$

we find the following expressions for the displacement and stress components for the region L_2:

$$u_r^2(r,z) = \int_0^\infty \{I_1(rs)C(s) + [4(1-\nu_2)I_1(rs) - rsI_0(rs)]D(s)\}\cos(sz)\,ds,$$

$$u_z^2(r,z) = \int_0^\infty [rsI_1(rs)D(s) - I_0(rs)C(s)]\sin(sz)\,ds,$$

$$\frac{\sigma_{rr}^2}{2\mu_2} = -\frac{1}{r}\int_0^\infty \{[I_1(rs) - rsI_0(rs)]C(s) + [(4-4\nu_2 + r^2s^2)I_1(rs)$$

$$-(3-2\nu_2)rsI_0(rs)]D(s)\}\cos(sz)\,ds,$$

$$\frac{\sigma_{zz}^2}{2\mu_2} = -\int_0^\infty s\{I_0(rs)C(s) - [2\nu_2I_0(rs) + rsI_1(rs)]D(s)\}\cos(sz)\,ds,$$

$$\frac{\sigma_{rz}^2}{2\mu_2} = -\int_0^\infty s\{I_1(rs)C(s) + [2(1-\nu_2)I_1(rs) - rsI_0(rs)]D(s)\}\sin(sz)\,ds, \tag{6}$$

where $I_p(\ )$ denotes the modified Bessel function of the first kind and of order $p \geqslant 0$.

3. Formulation of the problem

With reference to the cylindrical coordinate system chosen, the coaxial crack occupies the region $z = 0^{\pm}, r \geqslant a$ and the cylindrical inclusion is described by $r = b$ $(b < a)$, $-\infty < z < \infty$. The loading applied to the crack surface will be axially or rotationally symmetric. If the crack surfaces are subjected to equal and opposite normal tractions $p(r)$, then the boundary conditions at $z = 0$ can be written in the following form:

$$\sigma_{zz}^{1}(r, 0) = p(r), \qquad a \leqslant r < \infty, \tag{7}$$

$$u_z^1(r, 0) = 0, \qquad b < r < a, \tag{8}$$

$$\sigma_{rz}^{1}(r, 0) = 0, \qquad b < r < \infty, \tag{9}$$

$$\sigma_{rz}^{2}(r, 0) = 0, \qquad 0 < r < b, \tag{10}$$

$$u_z^2(r, 0) = 0, \qquad 0 < r < b. \tag{11}$$

Due to the continuity condition at the curved surface $r = b$, $|z| \geqslant 0$, we find that

$$\begin{aligned} u_z^1(b, z) &= u_z^2(b, z), \\ u_r^1(b, z) &= u_r^2(b, z), \\ \sigma_{rr}^1(b, z) &= \sigma_{rr}^2(b, z), \\ \sigma_{rz}^1(b, z) &= \sigma_{rz}^2(b, z). \end{aligned} \tag{12}$$

4. Reduction of the problem to a Fredholm integral equation of the second kind

We find with the help of the equations (4) and (6) that the boundary conditions (9), (10) and (11) are identically satisfied and the boundary conditions (7) and (8) lead to the dual integral equations

$$\int_0^\infty F(s) J_0(rs) ds + \int_0^\infty s\{[A(s) + 2\nu_1 B(s)] K_0(rs) - rsB(s) K_1(rs)\} ds$$
$$= \frac{-p(r)}{2\mu_1}, \qquad a \leqslant r < \infty, \tag{13}$$

$$\int_0^\infty \frac{1}{s} F(s) J_0(rs) ds = 0, \qquad b \leqslant r < a. \tag{14}$$

Let us assume

$$F(s) = s\int_a^\infty h(t) \cos(st) dt, \tag{15}$$

such that $h(\infty) = 0$. Using (15) we find that equation (14) is satisfied identically whatever be the form of $h(t)$. We can rewrite equation (13) in the form:

$$\frac{\partial}{\partial r} r\int_0^\infty \frac{F(s)J_1(rs)ds}{s} + r\int_0^\infty s\{[A(s)+2\nu_1 B(s)]K_0(rs) - rsK_1(rs)B(s)\}ds$$
$$= \frac{-p(r)r}{2\mu_1}, \qquad a \leqslant r < \infty. \qquad (16)$$

Now substituting the value of $F(s)$ from equation (15) into (16) we obtain

$$-\frac{\partial}{\partial r}\int_r^\infty \frac{th(t)dt}{(t^2-r^2)^{\frac{1}{2}}} + r\int_0^\infty s\{[A(s)+2\nu_1 B(s)]K_0(rs) - rsB(s)K_1(rs)\}ds$$
$$= \frac{-rp(r)}{2\mu_1}, \qquad a \leqslant r < \infty. \qquad (17)$$

The equation (17) is of Abel type, its solution may be written as

$$h(t) + \frac{2}{\pi}\int_0^\infty \left\{[A(s)+2\nu_1 B(s)]\int_t^\infty \frac{rK_0(rs)dr}{(r^2-t^2)^{\frac{1}{2}}} - sB(s)\int_t^\infty \frac{r^2K_1(rs)dr}{(r^2-t^2)^{\frac{1}{2}}}\right\}ds$$
$$= -\frac{1}{\pi\mu_1}\int_t^\infty \frac{rp(r)dr}{(r^2-t^2)^{\frac{1}{2}}}, \qquad a \leqslant t < \infty. \qquad (18)$$

With the aid of the formulae

$$\left.\begin{aligned} &\int_t^\infty \frac{rK_0(rs)dr}{(r^2-t^2)^{\frac{1}{2}}} = \frac{\pi}{2s}\, e^{-st}, \\ &\int_t^\infty \frac{r^2K_0(rs)dr}{(r^2-t^2)^{\frac{1}{2}}} = \frac{\pi}{2s^2}\,(1+st)e^{-st}, \end{aligned}\right\} \qquad (19)$$

we can rewrite the equation (18) in the form:

$$h(t) + \int_0^\infty [A(s) - (1-2\nu_1+st)B(s)]e^{-st}ds = -\frac{1}{\pi\mu_1}\int_t^\infty \frac{rp(r)dr}{(r^2-t^2)^{\frac{1}{2}}},$$
$$a \leqslant t < \infty. \qquad (20)$$

From the equations of continuity (12) we find that

$$\int_0^\infty \{[bsI_1(bs)D(s) - I_0(bs)C(s)] - [bsK_1(bs)B(s) - K_0(bs)A(s)]\}\sin(sz)ds$$
$$= \int_0^\infty \frac{1}{s}(2-2\nu_1+sz)F(s)J_0(bs)e^{-sz}ds, \qquad 0 < z < \infty \qquad (21)$$

$$\int_0^\infty \{[C(s)I_1(bs) + (4(1-\nu_2)I_1(bs)-bsI_0(bs))D(s)+K_1(bs)A(s)$$
$$-(4(1-\nu_1)K_1(bs)+bsK_0(bs))B(s)]\}\cos(sz)ds$$
$$=\int_0^\infty \frac{1}{s}(2\nu_1-1+sz)F(s)J_1(bs)e^{-sz}ds, \qquad 0<z<\infty, \tag{22}$$

$$\mu_1\int_0^\infty \{[K_1(bs)+bsK_0(bs)]A(s)-[(4(1-\nu_1)+b^2s^2)K_1(bs)$$
$$+(3-2\nu_1)bsK_0(bs)]B(s)\}\cos(sz)ds+\mu_2\int_0^\infty \{[I_1(bs)-bsI_0(bs)]C(s)$$
$$+[(4-4\nu_2+b^2s^2)I_1(bs)-(3-2\nu_2)bsI_0(bs)]D(s)\}\cos(sz)ds$$
$$=-\mu_1 b\int_0^\infty \left[(sz-1)J_0(bs)+(1-2\nu_1-sz)\frac{J_1(bs)}{bs}\right]F(s)e^{-sz}ds,$$
$$0<z<\infty, \tag{23}$$

$$\int_0^\infty [\mu_1 s\{K_1(bs)A(s)-(2(1-\nu_1)K_1(bs)+bsK_0(bs))B(s)\}+\mu_2 s\{I_1(bs)C(s)$$
$$+2(1-\nu_2)I_1(bs)-bsI_0(bs)D(s)\}]\sin(sz)ds$$
$$=z\mu_1\int_0^\infty sF(s)J_1(bs)e^{-zs}ds, \qquad 0<z<\infty. \tag{24}$$

With the help of the Fourier inversion theorem, we find from (21), (22), (23) and (24) that

$$bsI_1(bs)D(s)-I_0(bs)C(s)-bsB(s)K_1(bs)+K_0(bs)A(s)$$
$$=\frac{2}{\pi}\int_0^\infty \frac{F(u)}{u}(2(1-\nu_1)f_1+uf_2)J_0(bu)du=X_1, \qquad 0<s<\infty, \tag{25}$$

$$I_1(bs)C(s)+[4(1-\nu_2)I_1(bs)-bsI_0(bs)]D(s)+K_1(bs)A(s)$$
$$-[4(1-\nu_1)K_1(bs)+bsK_0(bs)]B(s)=\frac{2}{\pi}\int_0^\infty \frac{1}{u}((2\nu_1-1)f_3+uf_4)F(u)J_1(bu)du,$$
$$=X_2, \qquad 0<s<\infty, \tag{26}$$

$$\mu_1\{[K_1(bs)+bsK_0(bs)]A(s)-[(4(1-\nu_1)+b^2s^2)K_1(bs)+(3-2\nu_1)bsK_0(bs)]B(s)\}$$
$$+\mu_2\{[I_1(bs)-bsI_0(bs)]C(s)+[(4-4\nu_2+b^2s^2)I_1(bs)-(3-2\nu_2)bsI_0(bs)]D(s)\}$$
$$=-\frac{2}{\pi}\mu_1 b\int_0^\infty\left[(-f_3+uf_4)J_0(bu)+((1-2\nu_1)f_3-uf_4)\frac{J_1(bu)}{bu}\right]F(u)du$$
$$=X_3\mu_2, \qquad 0<s<\infty. \tag{27}$$

$$[\mu_1\{K_1(bs)A(s)-(2(1-\nu_1)K_1(bs)+bsK_0(bs))B(s)\}+\mu_2\{I_1(bs)C(s)$$

$$+(2(1-\nu_2)I_1(bs)-bsI_0(bs))D(s)\}=\frac{2}{\pi}\mu_1\int_0^\infty uF(u)f_2J_1(bu)du$$

$$=\frac{X_4\mu_2}{s}, \qquad 0<s<\infty, \tag{28}$$

where

$$f_1=\int_0^\infty \sin(sz)e^{-uz}dz=\frac{s}{s^2+u^2},$$

$$f_2=\int_0^\infty z\sin(sz)e^{-uz}dz=\frac{2su}{(s^2+u^2)^2},$$

$$f_3=\int_0^\infty \cos(sz)e^{-uz}dz=\frac{u}{(s^2+u^2)},$$

$$f_4=\int_0^\infty z\cos(sz)e^{-uz}dz=\left[\frac{u^2-s^2}{(u^2+s^2)^2}\right]. \tag{29}$$

In the rest of the analysis we shall denote $I_0(bs)=I_0, I_1(bs)=I_1, K_0(bs)=K_0, K_1(bs)=K_1$. Now solving the equations (25), (26), (27), and (28) we obtain

$$A=-\frac{1}{(a_2a_6-a_3a_5)(a_{12}I_0+a_7I_1)}[X_1\{(a_{13}I_0+a_8I_1)(a_1a_6-a_3a_4)+a_3I_1(a_{12}I_0+a_7I_1)\}$$

$$+X_2\{a_3I_0(a_{12}I_0+a_7I_1)+(a_9I_1+a_{14}I_0)(a_1a_6-a_3a_4)\}$$

$$+X_3\{(a_1a_6-a_3a_4)(a_{10}I_1+a_{15}I_0)-I_1a_6(a_{12}I_0+a_7I_1)\}$$

$$+X_4\{(a_{11}I_1+a_{16}I_0)(a_1a_6-a_3a_4)+\frac{a_6}{s}(I_0a_{12}+a_7I_1)(I_1-bsI_0)\}], \tag{30}$$

$$B=\frac{1}{(a_2a_6-a_3a_5)(a_{12}I_0+a_7I_1)}[X_1\{a_2I_1(a_{12}I_0+a_7I_1)-(a_2a_4-a_1a_5)(a_{13}I_0+a_8I_1)\}$$

$$+X_2\{a_2I_0(a_{12}I_0+a_7I_1)-(a_2a_4-a_1a_5)(I_1a_9+I_0a_{14})\}$$

$$-X_3\{a_5I_1(a_{12}I_0+a_7I_1)+(a_{10}I_1+a_{15}I_0)(a_2a_4-a_1a_5)\}$$

$$-X_4\{(a_2a_4-a_1a_5)(a_{11}I_1+a_{16}I_0)-\frac{a_5}{s}(I_1-bsI_0)(a_{12}I_0+a_7I_1)\}], \tag{31}$$

$$C=\frac{1}{I_1a_7+I_0a_{12}}[X_1(a_7a_{13}-a_8a_{12})+X_2(a_7a_{14}-a_{12}a_9)+X_3(a_{15}a_7-a_{10}a_{12})$$

$$+X_4(a_7a_{16}-a_{11}a_{12})], \tag{32}$$

$$D=\frac{1}{a_{12}I_0+a_7I_1}[X_1(a_{13}I_0+a_8I_1)+X_2(I_1a_9+I_0a_{14})+X_3(a_{10}I_1+a_{15}I_0)$$

$$+X_4(a_{11}I_1+a_{16}I_0)], \tag{33}$$

where

$$a_1 = b^2s^2(I_1^2-I_0^2)+2(1-\nu_2)I_1^2, \qquad a_2 = bsG[K_0I_1+K_1I_0],$$

$$a_3 = -G[K_1I_1(2-2\nu_1+b^2s^2)+2(1-\nu_1)bsI_0K_1+2bs(1-\nu_1)K_0I_1+K_0I_0b^2s^2],$$

$$a_4 = bs(I_1^2-I_0^2)+4(1-\nu_2)I_1I_0, \qquad a_5 = a_2/(bsG),$$

$$a_6 = -[bs(I_1K_1+K_0I_0)+4(1-\nu_1)K_1I_0],$$

$$a_7 = bsI_1+\frac{sbK_1(a_2a_4-a_5a_1)+K_0(a_3a_4-a_6a_1)}{a_2a_6-a_3a_5},$$

$$a_8 = 1+\frac{I_1(a_2sbK_1+a_3K_0)}{a_2a_6-a_3a_5}, \qquad a_9 = \frac{I_0(K_1bsa_2+K_0a_3)}{a_2a_6-a_3a_5},$$

$$a_{10} = -\frac{I_1(a_6K_0+bsa_5K_1)}{a_2a_6-a_3a_5}, \qquad a_{11} = \frac{a_6K_0+a_5bsK_1}{s}\ \frac{I_1-bsI_0}{a_2a_6-a_3a_5},$$

$$a_{12} = 2(1-\nu_2)I_1-bsI_0+\frac{GK_1(a_3a_4-a_1a_6)}{a_2a_6-a_3a_5}+\frac{G(a_2a_4-a_1a_5)}{a_2a_6-a_3a_5}[2(1-\nu_1)K_1+bsK_0],$$

$$a_{13} = \frac{G}{a_2a_6-a_3a_5}[a_3K_1I_1+a_2I_1\{2(1-\nu_1)K_1+bsK_0\}],$$

$$a_{14} = \frac{G}{a_2a_6-a_3a_5}[a_3I_0K_1+a_2I_0\{2(1-\nu_1)K_1+bsK_0\}],$$

$$a_{15} = \frac{-GI_1}{a_2a_6-a_3a_5}[a_6K_1+a_5\{2(1-\nu_1)K_1+bsK_0\}],$$

$$a_{16} = \frac{1}{s}\left[1+\frac{G(I_1-bsI_0)}{a_2a_6-a_3a_5}\{a_6K_1+[2(1-\nu_1)K_1+bsK_0]a_5\}\right] \tag{34}$$

and

$$G = \mu_1/\mu_2. \tag{35}$$

Substituting the values of $A(s)$ and $B(s)$ from equations (30) and (31) we can write

$$A(s)-(1-2\nu_1+st)B(s) = X_1B_1(s,t)+X_2B_2(s,t)+X_3B_3(s,t)+X_4B_4(s,t), \tag{36}$$

where

$$D_1 = -\frac{1}{(a_2a_6-a_3a_5)(a_{12}I_0+a_7I_1)}, \tag{37}$$

$$B_1(s,t) = D_1[(a_{13}I_0+a_8I_1)(a_1a_6-a_3a_4)+a_3I_1(a_{12}I_0+a_7I_1)$$
$$+(1-2\nu_1+st)\{a_2I_1(a_{12}I_0+a_7I_1)-(a_2a_4-a_1a_5)(a_{13}I_0+a_8I_1)\}], \tag{38}$$

$$B_2(s,t) = D_1[a_3I_0(a_{12}I_0 + a_7I_1) + (a_9I_1 + a_{14}I_0)(a_1a_6 - a_3a_4) + (1-2\nu_1 + st)\{a_2I_0(a_{12}I_0 + a_7I_1) - (a_2a_4 - a_1a_5)(I_1a_9 + I_0a_{14})\}], \tag{39}$$

$$B_3(s,t) = D_1[(a_{10}I_1 + a_{15}I_0)(a_1a_6 - a_3a_4) - I_1a_6(a_{12}I_0 + a_7I_1) - (1-2\nu_1 + st)(a_5I_1(a_{12}I_0 + a_7I_1) + (a_{10}I_1 + a_{15}I_0)(a_2a_4 - a_1a_5))], \tag{40}$$

$$B_4(s,t) = D_1[(a_{11}I_1 + a_{16}I_0)(a_1a_6 - a_3a_4) + \frac{a_6}{s}(a_{12}I_0 + a_7I_1)(I_1 - bsI_0) + (1-2\nu_1 + st)\{\frac{a_5}{s}(I_1 - bsI_0)(a_{12}I_0 + a_7I_1) - (a_2a_4 - a_1a_5)(a_{11}I_1 + a_{16}I_0)\}]. \tag{41}$$

From equation (25), we find that

$$X_1 = \frac{2}{\pi}\int_0^\infty \frac{F(u)}{u}[2(1-\nu_1)f_1 + uf_2]J_0(bu)du.$$

Making use of (15) and (29) we can write

$$X_1 = \frac{4(1-\nu_1)s}{\pi}\int_a^\infty h(t)dt\int_0^\infty \frac{J_0(ub)\cos(ut)du}{u^2+s^2} + \frac{4}{\pi}s\int_a^\infty h(t)dt\int_0^\infty \frac{u^2J_0(ub)\cos(ut)du}{(u^2+s^2)^2}. \tag{42}$$

Making use of the integrals $(62)_1$ and $(62)_3$ given in the Appendix, we find from (42) that

$$X_1 = [(3-2\nu_1)I_0(bs) + bsI_1(bs)]\int_a^\infty h(u)e^{-su}du - sI_0(bs)\int_a^\infty uh(u)e^{-su}du. \tag{43}$$

From (26) we find that

$$X_2 = \frac{2}{\pi}\int_0^\infty \frac{1}{u}[(2\nu_1 - 1)f_3 + uf_4]F(u)J_1(bu)du.$$

Making use of (15) and (29) we get

$$X_2 = \frac{2}{\pi}[(2\nu_1-1)\int_a^\infty h(t)dt\int_0^\infty \frac{u\cos(ut)J_1(bu)du}{u^2+s^2} + \int_a^\infty h(t)dt\int_0^\infty u\frac{u^2-s^2}{(u^2+s^2)^2}J_1(bu)\cos(ut)du]. \tag{44}$$

Using the integrals $(62)_4$ and $(62)_6$ we can write the equation (44) in the form

$$X_2 = -\left\{\frac{bs}{2}\,[I_0(bs)+I_2(bs)]+2\nu_1 I_1(bs)\right\}\int_a^\infty h(u)e^{-su}du$$
$$+\,sI_1(bs)\int_a^\infty uh(u)e^{-su}du\,. \tag{45}$$

From equations (27) and (45), we find that:

$$X_3 = -\frac{2}{\pi}\,bG\int_0^\infty\left[(-f_3+uf_4)J_0(bu)+\{(1-2\nu_1)f_3-uf_4\}\frac{J_1(bu)}{bu}\right]F(u)du\,. \tag{46}$$

Making use of (15), (29) and the integrals $(62)_3$, $(62)_4$ and $(62)_6$ we find that

$$X_3 = G\int_a^\infty h(t)e^{-st}[sb\,\{\tfrac{1}{2}I_0(sb)+bsI_1(bs)-stI_0(bs)\}$$
$$+\,I_1(bs)(st-2\nu_1)-\frac{sb}{2}\,I_2(bs)]\,dt\,. \tag{47}$$

From (28) we find the value of X_4 in the form

$$X_4 = \frac{2}{\pi}\,G\int_0^\infty uF(u)f_2(u)J_1(bu)du\,. \tag{48}$$

With the help of (15) and (29) we find that

$$X_4 = \frac{4}{\pi}\,sG\left[\int_a^\infty h(t)dt\left\{\int_0^\infty\frac{uJ_1(ub)\cos(ut)du}{s^2+u^2}\right.\right.$$
$$\left.\left.-\int_0^\infty\frac{us^2J_1(ub)\cos(ut)du}{(s^2+u^2)^2}\right\}\right]. \tag{49}$$

Making use of integrals $(62)_4$ and $(62)_5$ we find that

$$X_4 = -sG\left[2I_1(bs)+\frac{sb}{2}\,\{I_0(bs)+I_2(bs)\}\right]$$
$$\int_a^\infty h(u)e^{-su}du+s^2GI_1(bs)\int_a^\infty uh(u)e^{-su}du\,. \tag{50}$$

With the help of (43), (45), (47), (50) and (36) we can write equation (20) in the form

$$h(t)+\int_a^\infty h(u)K(u,t)du = -\frac{1}{\pi\mu_1}\int_t^\infty\frac{rp(r)dr}{(r^2-t^2)^{\frac{1}{2}}}\,,\qquad a<t<\infty\,, \tag{51}$$

where

$$K(u,t) = \int_0^\infty e^{-s(u+t)}[B_1(s,t)\{((3-2\nu_1)I_0(bs) + bsI_1(bs) - suI_0(bs))\}$$
$$+ B_2(s,t)\{-\frac{bs}{2}(I_0(bs)+I_2(bs)) - 2\nu_1 I_1(bs) + suI_1(bs)\}$$
$$+ GB_3(s,t)\{\frac{sb}{2}(I_0(bs) + 2bsI_1(bs) - 2suI_0(bs)) + I_1(bs)(su-2\nu_1) - \frac{sb}{2}I_2(bs)\}$$
$$+ GB_4(s,t)\{s(su-2)I_1(bs) - \frac{bs^2}{2}(I_0(bs)+I_2(bs))\}]ds. \tag{52}$$

The equation (51) is a Fredholm integral equation of the second kind with a kernel $K(u,t)$ defined by equation (52).

If $p(r) = -1/r^2$, then equation (51) can be written in the form

$$H(t) + \int_a^\infty H(u)K(u,t)du = \frac{1}{t}, \qquad a \leqslant t < \infty, \tag{53}$$

where

$$2\mu_1 h(u) = H(u). \tag{54}$$

5. Expression for the stress-intensity factor

We find from equation (4) that

$$\frac{\sigma_{zz}^1(r,0)}{2\mu_1} = -\int_0^\infty F(s)J_0(rs)ds - \int_0^\infty s\{K_0(rs)A(s)$$
$$+ [2\nu_1 K_0(rs) - rsK_1(rs)]B(s)\}ds, \qquad 0 < r < a, \tag{55}$$

which may be written in an alternative form:

$$\sigma_{zz}^1(r,0) = 2\mu_1\left[\frac{h(a)}{(a^2-r^2)^{\frac{1}{2}}} + \int_a^\infty \frac{h'(t)dt}{(t^2-r^2)^{\frac{1}{2}}}\right.$$
$$-\int_0^\infty s\{[A(s) + 2\nu_1 B(s)]K_0(rs)$$
$$\left.- rsK_1(rs)B(s)\}ds\right], \qquad 0 < r < a, \tag{56}$$

where the prime denotes differentiation with respect to t. The stress-intensity factor at the tip of the crack is given by the equation

$$K = \lim_{r\to a^+}[\sqrt{2(a-r)}\,\sigma_{zz}^1(r,0)]. \tag{57}$$

If in equation (57) we use the expression (56) for $\sigma'_{zz}(r, 0)$ with the values (30), (31) substituted for $A(s)$ and $B(s)$ we find that only the first term in (56) makes contributions to the limit in (57) and that:

$$K = \frac{2\mu_1 h(a)}{(a)^{\frac{1}{2}}} = \frac{H(a)}{(a)^{\frac{1}{2}}}\ . \tag{58}$$

For an infinite solid with an external circular crack subjected to rotationally symmetric loading ($b = 0$) the stress-intensity factor is given by (see Kassir and Sih [2], p. 41):

$$K_\infty = \frac{2}{\pi a^{\frac{1}{2}}} \int_a^\infty \frac{rp(r)dr}{(r^2 - a^2)^{\frac{1}{2}}}\ . \tag{59}$$

For $p(r) = -1/r^2$, we find that

$$K_\infty = \frac{1}{a^{3/2}}\ . \tag{60}$$

Now from (58) and (60), we have

$$\frac{K}{K_\infty} = aH(a)\,. \tag{61}$$

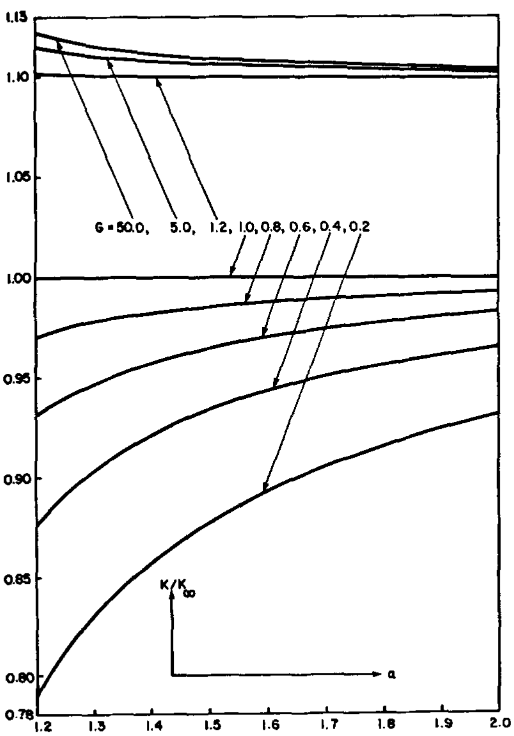

Figure 1. Values of K/K_∞ against a for various values of G and $\nu_1 = \nu_2 = 0.25$, $b = 1$.

6. Numerical results

We have solved the integral equation (53) numerically for $H(t)$ for $\nu_1 = \nu_2 = \frac{1}{4}$, $b = 1$ and $a = 1.2$ (0.1) 2.0 for various values of G. These numerical values of $H(a)$ are used to obtain the numerical values of K/K_∞. In Figure 1 the numerical values of K/K_∞ are displayed graphically against a for various values of G.

Appendix

We shall list here some of the integrals, which have been used in the body of the paper (see Erdelyi [5], Vol. 1) for $t > b$:

$$\int_0^\infty \frac{J_0(ub)\cos(ut)du}{u^2+s^2} = \frac{\pi}{2s} I_0(bs)e^{-st},$$

$$\int_0^\infty \frac{J_0(ub)\cos(ut)du}{(s^2+u^2)^2} = \frac{\pi e^{-st}}{4s^3}\,[I_0(bs) + stI_0(bs) - bsI_1(bs)],$$

$$\int_0^\infty \frac{u^2 J_0(ub)\cos(ut)du}{(s^2+u^2)^2} = \frac{\pi}{4s}\, e^{-st}\,[I_0(bs) + bsI_1(bs) - stI_0(sb)],$$

$$\int_0^\infty \frac{u\cos(ut)J_1(bu)du}{u^2+s^2} = -\frac{\pi}{2}\, e^{-st} I_1(bs),$$

$$\int_0^\infty \frac{u\cos(ut)J_1(bu)du}{(u^2+s^2)^2} = -\frac{\pi e^{-st}}{8s}\,[2tI_1(bs) - b\{I_0(bs) + I_2(bs)\}],$$

$$\int_0^\infty u\left(\frac{u^2-s^2}{(u^2+s^2)^2}\right) J_1(bu)\cos(ut)du = \frac{\pi}{4}\, e^{-st}\,[2I_1(bs)(st-1) - sb\{I_0(bs) + I_2(bs)\}]. \tag{62}$$

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